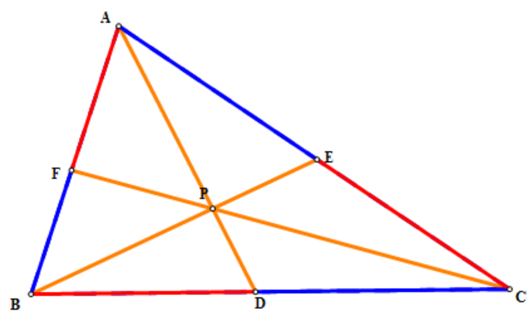
Investigation of Triangles

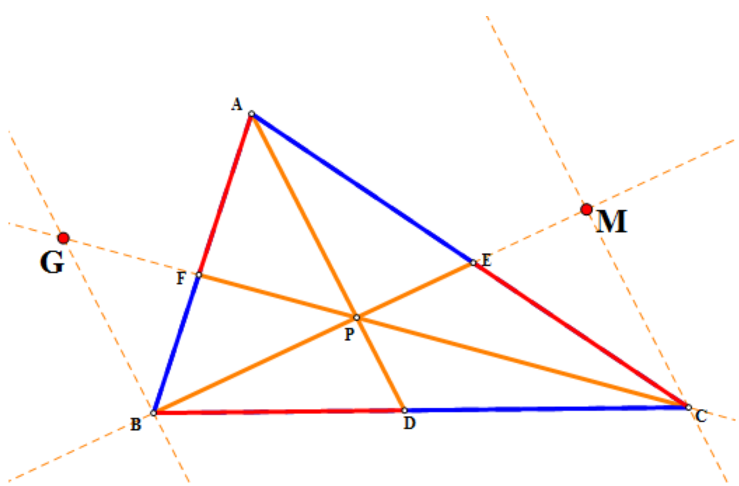
Problem 1: Consider a triangle ABC. Select point P inside the triangle and draw lines AP, BP, and CP then extend them to their intersections with the opposite sides in points D, E, and F respectively.



Click on GSP1 file to explore further.

Prove that, $\frac{DC \cdot AE \cdot BF}{AF \cdot CE \cdot BD} = 1$, regardless of the triangle or position of P.

Proof: First we construct two auxiliary lines parallel to the segment AD through the vertices B and C. Then extend the lines FC and BE to intersect these parallel lines at G and M respectively.



Examining the triangle, we find the following similarities:

$\Delta PDC \sim \Delta GBC$ and $\Delta PDB \sim \Delta MCB$. So,

$\frac{PD}{DC} = \frac{BG}{BC}$ .... (i) and $\frac{PD}{BD} = \frac{MC}{BC}$ ... ... (ii)

Solving equation (i), $PD = \frac{DC \cdot BG}{BC}$

Now, substituting PD in equation (ii), we get: $MC = \frac{DC \cdot BG}{BD} \dots \dots \dots (iii)$

From the diagram above we can also see that $\Delta BGF \sim \Delta APF$ (by AA) and $\Delta MCE \sim \Delta PAE$.

Hence, we have, $\frac{MC}{CE} = \frac{PA}{AE}$ and $\frac{BG}{BF} = \frac{PA}{AF}$.

Solving the equation for MC and BG respectively yields,

$BG = \frac{BF \cdot PA}{AF}$ and $MC = \frac{CE \cdot PA}{AE}$

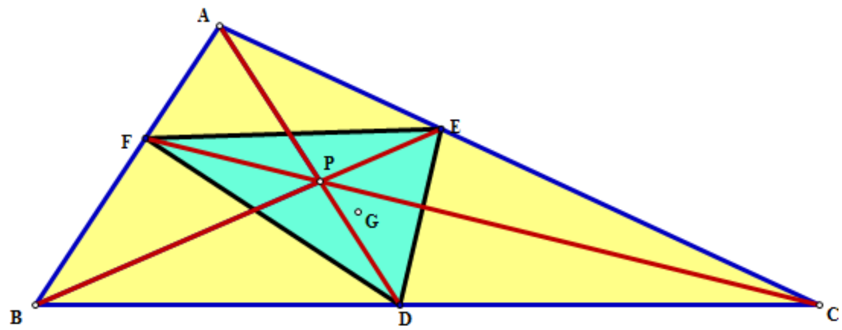
Now, substituting BG and MC in equation (iii), we get

$$\frac{DC \cdot BF}{AF \cdot BD} = \frac{CE}{AE}$$

Now, multiplying both sides by $\frac{AE}{CE}$:

$$\frac{DC \cdot AE \cdot BF}{AF \cdot CE \cdot BD} = 1$$

Problem 2: Show that when P is inside triangle ABC, the ratio of the area of triangle ABC and triangle DEF is always greater than or equal to 4. When is it equal to 4?



Click on GSP2 file for further exploration.

Proof: Here P is the Centroid of the triangle ABC. Hence line segments AD, CF, and BE are the medians of triangle ABC and D, E, and F are the midpoints of sides BC, AC, and AB respectively.

Hence, we can write, BD = DC = a, CE = EA = b, and AF = FB = c.

From Triangle mid-segment theorem, a segment joining the midpoints of any two sides will be parallel to the third side and half its length. So, FE = a, DF = b, and FD = c.

Using Heron's formula, the area of triangle DEF = $\sqrt{s(s-a)(s-b)(s-c)}$, where s= $\frac{a+b+c}{2}$.

The semiperimeter of triangle ABC is $\frac{2a+2b+2c}{2} = a + b + c = 2s$

Using Heron's formula to find the area of triangle ABC:

$$A = \sqrt{2s(2s-2a)(2s-2b)(2s-2c)}$$

$$A = \sqrt{16s(s-a)(s-b)(s-c)}$$

$$A = 4\sqrt{s(s-a)(s-b)(s-c)}$$

Hence, when P is the Centroid of triangle ABC, the area of triangle ABC is 4 times the area of triangle DEF.